Homework #12

For Friday, April 15

⋆ 1. Assuming $\theta(x)$ and η are any formulas and x is not free in η, prove $\exists x\ \theta(x) \to \eta$ from $\forall x\ (\theta(x) \to \eta)$.

⋆ 2. Suppose φ and $\psi(x)$ are any formulas, and x is not free in φ. Prove

$$(\varphi \to \exists x\ \psi(x)) \leftrightarrow \exists x\ (\varphi \to \psi(x))$$

using the following steps:

a. First prove the $\leftarrow$ direction.

b. From $\exists x\ \psi(x)$, prove $\exists x\ (\varphi \to \psi(x))$.

c. From $\neg\exists x\ \psi(x)$ and $\varphi \to \exists x\ \psi(x)$, conclude $\exists x\ (\varphi \to \psi(x))$. (Hint: from the hypotheses, show that φ implies *anything*.)

d. Put parts (b) and (c) together with a proof of $\exists x\ \psi(x) \vee \neg\exists x\ \psi(x)$ (you don't have to write out the latter) to obtain a proof the $\to$ direction.

Note that this $\to$ direction of this problem, together with problem 11, are used in the proof of van Dalen's Lemma 3.1.7.

⋆ 3. Let L_1, L_2, and L_3 be languages, with $L_1 \supseteq L_2 \supseteq L_3$. (In other words, L_1 has all the constant, function, and relation symbols of L_2, and possibly more; and similarly for L_2 and L_3.) Suppose T_1, T_2, and T_3 are theories in the languages L_1, L_2, and L_3 respectively. Show that if T_1 is a conservative extension of T_2, and T_2 is a conservative extension of T_3, then T_1 is a conservative extension of T_3. (You can find the definition of "conservative extension" on page 104 in van Dalen, Definition 3.1.5.)

4. Consider the following two statements of the completeness theorem:

> *Version A:* If Γ is any consistent set of sentences, then Γ has a model.
>
> *Version B:* If Γ is any set of sentences, φ is any sentence, and $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.

Show *directly* that these two statements are equivalent, i.e. that each one implies the other. (Hint: to show that B implies A, take φ to be $\perp$.)